Nakamaye's theorem.

X is a smooth projective variety of dimension n / $\mathbb{C}$.

Let L be a divisor with $\kappa(L) \geq 0$.

Define $\mathbb{B}(L) := \bigcap_{m \geq 1} Bs(|mL|)$ Stable base locus of L

$\mathbb{B}(L) = Bs(|pL|)$ for $p >> 0$

<u>Defn</u>: The augmented base locus of L is the Zariski-closed set

$$\mathbb{B}_+(L) := \mathbb{B}(L - \varepsilon A)$$

for any ample A and $0 < \varepsilon << 1$.

<u>Defn</u>: Given a nef and big divisor L on X, the null locus $\mathrm{Null}(L)$ of L is the union of all positive dimensional subvarieties $V \subseteq X$ with

$$(L^{\dim V} \cdot V) = 0.$$

Theorem [Nakamaye 2000]

If L is any big and nef divisor on X, then

$$\mathbb{B}_+(L) = \mathrm{Null}(L).$$

Sketch:

- Realize $\mathbb{B}_+(L)$ as the zero-locus of a multiplier ideal, by constructing a divisor with high multiplicity on $\mathbb{B}_+(L)$, and multiplicity less than one outside
- Use Nadel's vanishing to get that the restriction of sections from X to the subscheme defined by the multiplier ideal is surjective.
- Study an irreducible component of $\mathbb{B}_+(L)$ using the primary decomposition of multiplier ideals. Using R-R one can compare the dimensions of cohomological groups to get the result.

Proposition: Let L be a big divisor, A an ample $\mathbb{Q}$-divisor on X. Then the stable base locus $\mathbb{B}(L-\varepsilon A)$ is independent of ε, provided $0<\varepsilon\ll 1$.

If A' is a second ample divisor, then $\mathbb{B}(L-\varepsilon A)=\mathbb{B}(L-\varepsilon' A')$ for $0<\varepsilon,\varepsilon'\ll 1$.

Finally, these loci only depend on the numerical class of L.

Proof:

- If $0<\varepsilon_1<\varepsilon_2$, then $\mathbb{B}(L-\varepsilon_1 A)\subseteq \mathbb{B}(L-\varepsilon_2 A)$
- Fix $\delta>0$ s.t. $A''=A-\delta A'$ is ample.

$$\mathbb{B}(L-\varepsilon A)=\mathbb{B}(L-\varepsilon\delta A'-\varepsilon A'')\supseteq \mathbb{B}(L-\varepsilon\delta A')$$

- Given $P\equiv 0$, take $A'=A-\frac{1}{\varepsilon}P$. Then $L+P-\varepsilon A=L-\varepsilon A'$

Lemma: Let L be a big and nef on X. Then $\mathrm{Null}(L)$ is a Zariski-closed subset of X, and every irreducible component V of $\mathrm{Null}(L)$ satisfies

$$(L^{\dim V} \cdot V) = 0.$$

Proof:

Let V be the closure of the union of $\{V_i\}_{i \in I}$ with $V_i \subseteq \mathrm{Null}(L)$

We need to show that $V \subseteq \mathrm{Null}(L)$.

Suppose that $(L^{\dim V} \cdot V) > 0$, so $\mathcal{O}_V(L)$ is big. Then for an ample divisor A in V, there is $p \gg 0$ s.t. $\mathcal{O}_V(pL - A)$ has a non-vanishing section. Take W to be the zero-locus of that section.
Take $T \not\subseteq W$, Then $\mathcal{O}_T(pL-A)$ has a non-van section, so $\mathcal{O}_T(L)$ is big.
Hence all $V_i \subseteq W$, contradicting that V is the closure.

Proof of Nakamaye's theorem:

- $\mathrm{Null}(L) \subseteq \mathbb{B}_+(L)$.

If $(L^{\dim V} \cdot V) = 0$, then $L|_V$ is on the boundary of $\overline{\mathrm{Eff}}(V)$, so given any ample divisor of X and any $\varepsilon > 0$, $(L - \varepsilon A)|_V$ is outside the $\overline{\mathrm{Eff}}(V)$.
If $V \not\subseteq \mathbb{B}(L - \varepsilon A)$, then $\exists$ effective divisor $D \sim mL - kA$ s.t. $V \not\subseteq \mathrm{Supp}(D)$.
Hence $mL - kA|_V \in \overline{\mathrm{Eff}}(V)$,

- Realize $\mathbb{B}_+(L)$ as the zero-locus of a multiplier ideal.

Fix A very ample s.t. $A - K_X$ is ample. Choose $a, p \gg 0$ s.t.

$$\mathbb{B}_+(L) = \mathbb{B}(aL - 2A) = Bs(|paL - 2pA|).$$

Choose $n+1$ general divisors $E_1, \dots, E_{n+1} \in |paL - 2pA|$, and define

$$D = \frac{n}{n+1}(E_1 + \dots + E_{n+1})$$

Then $\mathrm{mult}_x(D) \geq n$ if $x \in \mathbb{B}_+(L)$, and if $x \in X \setminus \mathbb{B}_+(L)$ we have that $\mathcal{J}(D)$ is trivial.

Hence $\mathrm{Zeroes}(\mathcal{J}(D)) = \mathbb{B}_+(L)$ set-theo.

- Use Nadel's vanishing.

Set $q = np$. We have that $D \equiv qaL - 2qA$.

Because L is nef and $A - K_X$ is ample, Nadel vanishing implies that

$$H^1(X, \mathcal{O}_X(mL - qA) \otimes \mathcal{J}(D)) = 0 \quad \text{for } m \geq qa.$$

Write $Z \subseteq X$ the subscheme defined by $\mathcal{J}(D)$.
Then

$$H^0(X, \mathcal{O}_X(mL - qA)) \to H^0(Z, \mathcal{O}_Z(mL - qA))$$

is surjective for $m \geq qa$.

- $\mathbb{B}_+(L) = \mathrm{Bs}(|mL - qA|)$ for $m \geq qa$

Notice that $(m-qa)L + qA$ is g.g., because it is 0-regular (Mumford regularity)
This follows from Kodaira vanishing

$$(m-qa)L + (q-i)A = K_X + \underbrace{A - K_X}_{\text{ample}} + \underbrace{(m-qa)L}_{\text{nef}} + \underbrace{(q-i-1)A}_{\text{ample.}}$$

$$(m-qa)L + qA = (mL - qA) - (qaL - 2qA)$$

$$\begin{aligned}\mathrm{Bs}(|mL-qA|) &\subseteq \mathrm{Bs}(|qaL - 2qA|) \\ &= \mathrm{B}(aL - 2A) = \mathbb{B}_+(L)\end{aligned}$$

- Assume $V \subseteq \mathbb{B}_+(L)$ and $V \not\subseteq \mathrm{Null}(L)$.

irreducible

This means tha $\mathcal{O}_V(mL)$ is big and nef, and also $V \subseteq \mathrm{Bs}(|mL - qA|)$ for $m \gg 0$.

It is enough to show that, for $m \gg 0$, the restriction map

$$H^0(Z, \mathcal{O}_Z(mL - qA)) \longrightarrow H^0(V, \mathcal{O}_V(mL - qA))$$

is non-zero.

This produces a section $s \in H^0(X, \mathcal{O}_X(mL - qA))$ s.t. $s|_V \neq 0$.

- For any fixed divisor M on X, the restriction map
$$H^0(Z, \mathcal{O}_Z(mL+M)) \to H^0(V, \mathcal{O}_V(mL+M))$$
is non-zero for $m \gg 0$.

Take a primary decomposition of $\mathcal{I}_Z = \mathcal{J}(D)$. We want to construct $Y, W \subseteq X$ with $Y_{red} = V$, $\mathcal{I}_Z = \mathcal{I}_Y \cap \mathcal{I}_W$ and $V \cap W_{red} \subsetneq V$.

Take Y to be the $\mathcal{I}_V$-primary component of $\mathcal{I}_Z$, and $\mathcal{I}_W$ the intersection of the rest of the ideals,

We have the SES
$$0 \to \mathcal{O}_Z \to \mathcal{O}_Y \oplus \mathcal{O}_W \to \mathcal{O}_{Y \cap W}$$
$$(s, 0) \longmapsto 0$$
Twist by $\mathcal{O}_X(mL+M)$, it is enough to produce a section
$$s \in \ker\left(H^0(Y, mL+M) \to H^0(Y \cap W, mL+M)\right) =: K_m$$
such that s_{red} is non-vanishing

$$K_m' := \ker\left(H^0(Y, mL+M) \longrightarrow H^0(V, mL+M)\right)$$

We need to show that $K_m \not\subseteq K'_m$.

Note that $h^0(Y \cap W, mL+M)$ grows at most like $m^{\dim Y \cap W}$, so the codimension of K_m in $H^0(Y, mL+M)$ is $O(m^{\dim V - 1})$

On the other hand, consider

$$0 \to \mathcal{I}_{V/Y}(mL+M) \to \mathcal{O}_Y(mL+M) \to \mathcal{O}_V(mL+M) \to 0$$

$\mathcal{O}_V(L)$ is big and nef, so $h^0(V, mL+M) \geq C \cdot m^{\dim V}$

Also, $h^1(Y, \mathcal{I}_{V/Y}(mL+M)) = O(m^{\dim V - 1})$, so codim $K'_m = C \cdot m^{\dim V}$

Therefore K'_m cannot contain K_m. $\blacksquare$

Asymptotic R-R + L nef

$$h^i(Y, \mathcal{F}(mL)) = O(m^{\dim Y - i})$$